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ON A CERTAIN (MOD 2) IDENTITY AND A METHOD OF PROOF BY EXPANSION

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Beauty is often the child of imperfection

ABSTRACT. We prove the congruence

$$\prod_{\substack{n=1 \\ n\not\equiv 7 \pmod{14}}}^{\infty} (1-x^n) \equiv \sum_{-\infty}^{\infty} (x^{n(3n+2)} + x^{7n(3n+2)+2}) \pmod 2$$

by first establishing a related equation, which reduces to the congruence modulo 2. The method of proof (called "expanding zero") is based on a formula of the authors for expanding the product of two triple products. A second proof of the result more fully explicates the various aspects of the method. A parity result for an associated partition function is also included.

1. INTRODUCTION

In two previous papers [1], [2] we proved a collection of identities which had initially been discovered as (mod 2) congruences during a computer search, but which were later found to be equations over the integers when certain signs in them were changed. All these congruences led to parity theorems for the corresponding partition functions. (Cf. [3, Table 2].)

In this paper we prove another (mod 2) identity, discovered during the above-mentioned computer search, viz:

Theorem 1. *There holds*

$$(1) \qquad \prod_{\substack{n=1 \\ n\not\equiv 7 \pmod{14}}}^{\infty} (1-x^n) \equiv \sum_{-\infty}^{\infty} (x^{n(3n+2)} + x^{7n(3n+2)+2}) \pmod 2.$$

(Throughout this paper congruences will be understood to be (mod 2).)

Unlike the (mod 2) identities in [1] and [2], however, we found that this congruence had no equation over the integers of the same form standing behind

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it. (This conclusion was reached after an exhaustive search failed to find α_n, β_n, and ε_n equal to ± 1 for which the equation

$$\sum_{-\infty}^{\infty}(\alpha_n x^{n(3n+2)} + \beta_n x^{7n(3n+2)+2}) = \prod_{\substack{n=1 \\ n\not\equiv 7 \pmod{14}}}^{\infty} (1+\varepsilon_n x^n)$$

holds. (Cf. [2, §6].) However, since the identities proved in [1] and [2] were of interest in themselves and easily implied the mod 2 results, we take as our first task here the discovery of an equation that is equivalent to (1) modulo 2. This is accomplished in §2, where the desired equation (9) is derived. Also presented in §2 is the partition parity result associated with (1) (Theorem 3). The rest of the paper is then devoted to proving (9). (Actually we present two proofs and a discussion of the methods employed.)

The first proof is given in two parts in §3. The first part contains a transformation of (9) into equation (16) by means of the expansion formula developed in [2] and first used there. The second part contains the transformation of equation (16) into equation (17), which in turn is changed into the final equation (20). Equation (20) is then proved using a device we call an "expansion of zero". Since all of these steps are reversible, we have that (9) $\leftrightarrow$ (16) $\leftrightarrow$ (17) $\leftrightarrow$ (20).

In §4 we derive a further expansion formula and discuss other expansions of zero. The second proof of (9) is then presented in §5. Although this proof employs the same ideas as the proof of (20), it establishes (9) without the use of the quintuple product. Since the second proof is more elaborate, we follow it in §6 with a discussion of the ideas and analytic methods which were used in constructing that proof, such as the two algorithms, *Forward* and *Backward*, that allow us to discover which expansions are available to use. The second proof was primarily presented here to show that a proof of equation (9) (and similar identities) could be found in a somewhat systematic way and that the large number of terms which were generated by the expansions in the proof could readily be managed by using a computer.

2. The derivation of an equation that implies congruence (1)

We begin by recalling some material from [2, pp. 302-303]. Let $r_1, \dots, r_t$ be distinct residues modulo m and let $S = \{n \in Z^+ : n \equiv r_1, \dots, r_t \pmod{m}\}$. Then $(r_1, \dots, r_t)_m$ will denote the infinite product $\prod_{n\in S}(1-x^n)$ and $[r_1, \dots, r_t]_m$ will denote $\prod_{n\in S}(1+x^n)$.

For $\delta, \varepsilon \in \{0, 1\}$, we define the four one-variable T-functions by the formula

$$(2)\qquad \begin{aligned} T_{2\delta+\varepsilon}(k, l) &\stackrel{\text{def}}{=} \sum_{-\infty}^{\infty}(-1)^{\delta\frac{n(n+1)}{2}-n\varepsilon} x^{kn^2+ln} \\ &= \prod_{n=1}^{\infty}[1-(-1)^{n\delta}x^{2kn}][1+(-1)^{n\delta+\varepsilon}x^{2kn-k+l}] \\ &\quad \cdot [1+(-1)^{(n+1)\delta+\varepsilon}x^{2kn-k-l}], \end{aligned}$$

the Jacobi triple products being expressed concisely in the above notation as

$$(3)\qquad \begin{aligned} T_0(k, l) &= (0)_{2k}[\pm(k-l)]_{2k}, \\ T_1(k, l) &= (0, \pm(k-l))_{2k}, \\ T_2(k, l) &= (0, \pm(k+l))_{4k}[\pm(k-l), 2k]_{4k}, \\ T_3(k, l) &= (0, \pm(k-l))_{4k}[\pm(k+l), 2k]_{4k}. \end{aligned}$$

(For simplicity we usually write T for T_0.)

We also have the Gauss formula [2, (13)]

$$(4)\qquad \frac{1}{2}T\left(\frac{n}{2}, \frac{n}{2}\right) = \frac{(0)_{2n}}{(n)_{2n}} = T(2n, n).$$

To begin the derivation, we use (2) in (1), obtaining

$$\prod_{\substack{n=1 \\ n\not\equiv 7 \pmod{14}}}^{\infty} (1-x^n) \equiv \sum_{-\infty}^{\infty}(-1)^n x^{3n^2+2n} + x^2\sum_{-\infty}^{\infty}(-1)^n x^{21n^2+14n}$$

$$= T_1(3, 2) + x^2 T_1(21, 14).$$

Writing all the terms in product form gives

$$(5)\qquad (0, \pm1)_6 + x^2(0, \pm7)_{42} \equiv \frac{(0)_1}{(7)_{14}}.$$

Next, by [3, Ex. 4] and [2, (18)], we have

$$(\pm1)_6 \equiv \frac{1}{(\pm1)_3(0)_6(6)_{12}} \equiv \frac{1}{(\pm1)_3}$$

and its companion (by $x \to x^7$)

$$(\pm7)_{42} \equiv \frac{1}{(\pm7)_{21}}.$$

Thus, (5) becomes

$$\frac{(0)_6}{(\pm1)_3} + x^2\frac{(0)_{42}}{(\pm7)_{21}} \equiv \frac{(0)_1}{(7)_{14}} = \frac{(0)_{14}(0)_1^2}{(0, 7)_{14}(0)_1}$$

$$\equiv \frac{(0)_{14}(0)_2}{(0)_7(0)_1} = \frac{(0)_{14}(0)_2}{(0)_{21}(\pm7)_{21}(0)_3(\pm1)_3}.$$

Clearing fractions gives

$$(6)\qquad (0)_{21}(\pm7)_{21}(0)_3(0)_6 + x^2(0)_{42}(0)_{21}(0)_3(\pm1)_3 \equiv (0)_{14}(0)_2.$$

By [2, (18)], we have $(0)_3 \equiv \frac{1}{(3)_6}$, and so $(0)_{21} \equiv \frac{1}{(21)_{42}}$. Thus, (6) becomes

$$(7)\qquad (0)_{21}(\pm7)_{21}\frac{(0)_6}{(3)_6} + x^2\frac{(0)_{42}}{(21)_{42}}(0)_3(\pm1)_3 \equiv (0)_{14}(0)_2.$$

As this congruence stands, it is not an equation, which can readily be determined by comparing the power series expansions of the terms on the two sides. But one finds, after numerous trial sign changes, that the following is an equation:

$$(0)_{21}[\pm 7]_{21}\frac{(0)_6}{(3)_6} - x^2\frac{(0)_{42}}{(21)_{42}}(0)_3[\pm 1]_3 = (0, \pm 14)_{42}(0, \pm 2)_6. \tag{8}$$

(Working with power series, one finds that the two sides of (8) agree up to degree 1000, which implies heuristically that (8) is an equation.) Using (3) and (4), we can express (8) in terms of T-functions, thereby obtaining an equation of the kind we wished to derive:

Theorem 2. *We have*

$$T\left(\tfrac{21}{2}, \tfrac{7}{2}\right) T\left(\tfrac{3}{2}, \tfrac{3}{2}\right) - x^2 T\left(\tfrac{21}{2}, \tfrac{21}{2}\right) T\left(\tfrac{3}{2}, \tfrac{1}{2}\right) = 2T_1(21, 7)T_1(3, 1). \tag{9}$$

Two proofs of Theorem 2 are given respectively in §§3 and 5. It is also clear from the way that (9) was derived from (1) that congruence (1) will be true if equation (9) is shown to be true.

Before proceeding to the proof of Theorem 2, we will give two consequences of Theorem 1. Recall the definition of the Q-function and the Quintuple Product formula in our notation [2, p. 304], viz.

$$\begin{aligned} Q(m, k) &\stackrel{\text{def}}{=} \sum_{-\infty}^{\infty} x^{\frac{m(3n^2+n)}{2}}(x^{-3kn} - x^{3kn+k}) \\ &= T\left(\tfrac{3m}{2}, \tfrac{m}{2} - 3k\right) - x^k T\left(\tfrac{3m}{2}, \tfrac{m}{2} + 3k\right) \\ &= (0, \pm k, \pm(m-2k), \pm(m-k), m)_{2m}. \end{aligned} \tag{10}$$

Corollary. *Let* $S = \{n \in Z^+ : n \not\equiv 21 \pmod{42}\}$. *Then*

$$x\prod_{n\in S}(1 - x^n) \equiv \sum_{-\infty}^{\infty}(x^{(3n+1)^2} + x^{7(3n+1)^2}) \equiv Q(6, 1) + Q(42, 7).$$

Proof. Replacing x by x^3 in (1) and then multiplying by x gives the first congruence. Then (10) gives

$$Q(6, 1) = \sum_{-\infty}^{\infty} x^{9n^2} - \sum_{-\infty}^{\infty} x^{(3n+1)^2} \equiv 1 + \sum_{-\infty}^{\infty} x^{(3n+1)^2},$$

from which the second congruence follows. □

Theorem 3. *Let* $S = \{n \in Z^+ : n \equiv \pm(1, 3, 5, 9, 11, 13), 14 \pmod{28}\}$ *and let* $p(S; k)$ *denote the number of partitions of* k *whose parts lie in* S. *Then* $p(S; k)$ *is odd if and only if* $k = 3n^2 \pm n$ *or* $21n^2 \pm 7n + 2$, $n \geq 0$.

Proof. Using the algorithm in [3, §4] for finding the Euler reciprocal (mod 2), we discover that

$$\frac{1}{(0, \pm1, \pm2, \pm3, \pm4, \pm5, \pm6)_{14}} \equiv \frac{(0, \pm2, \pm4, \pm6)_{14}(\pm1, \pm3, \pm5, 7)_{14}^2}{(0, \pm1, \pm2, \pm3, \pm4, \pm5, \pm6)_{14}}$$
$$= (\pm1, \pm3, \pm5)_{14}(7)_{14}^2 \equiv (\pm1, \pm3, \pm5, \pm9, \pm11, \pm13, 14)_{28} \pmod 2.$$

Thus, congruence (1) implies

$$1 + \sum_{k=1}^{\infty} p(S; k)x^k = \prod_{n\in S} \frac{1}{1-x^n} \equiv \prod_{\substack{n=1 \\ n\not\equiv 7 \pmod{14}}}^{\infty} (1-x^n)$$
$$\equiv 1 + \sum_{n=1}^{\infty} (x^{n(3n\pm1)} + x^{7n(3n\pm1)+2}),$$

from which the result follows. □

3. The first proof of Theorem 2

We begin this section by observing that each term of (9) is the product of two T's—two terms with $T_0 \cdot T_0$ and the third with $T_1 \cdot T_1$. Because of the importance of an equation with this form, we introduce the following terminology.

Definition. A set of terms, or an equation, will be called a "T^2 set" or a "T^2 equation", respectively, if each of its nonzero terms has the form $ax^{\alpha}T_{\varepsilon_1} \cdot T_{\varepsilon_2}$, where a is a constant and $\varepsilon_1, \varepsilon_2 \in \{0, 1, 2, 3\}$. (Our use of "$T$" here is not short for "T_0".) Corresponding terminology will also be used for Q^2 sets and Q^2 equations.

In [2, p. 306] we developed the following formula for expanding the single-variable product $T_{\varepsilon_1}(k_1, l_1)T_{\varepsilon_2}(k_2, l_2)$, $\varepsilon_1, \varepsilon_2 \in \{0, 1\}$, into a finite sum of terms which form a T^2 set. We shall refer to this as "the expansion formula". It will be used extensively throughout the rest of the paper.

Expansion Formula. *Let* $a, b, m \in Z^+$ *and* $(k_1, l_1), (k_2, l_2) \in \{(\frac{i}{2}, \frac{j}{2}) : (i, j) \in Z^+ \times Z, i \equiv j \pmod 2\}$ *and assume* $\varepsilon_1, \varepsilon_2 \in \{0, 1\}$. *If the separability condition*

$$(11) \qquad k_1 b = k_2 a(m - ab)$$

is satisfied, then

$$T_{\varepsilon_1}(k_1, l_1)T_{\varepsilon_2}(k_2, l_2) = \sum_{r\in R} (-1)^{\varepsilon_2 r} x^{k_2 r^2 + l_2 r} T_{\delta_1}(K_1, L_1(r))T_{\delta_2}(K_2, L_2(r)),$$

where

$$K_1 = k_1 + k_2 a^2, \qquad K_2 = k_2 m(m - ab),$$

$$(12) \qquad L_1(r) = l_1 - l_2 a - 2k_2 ar, \qquad L_2(r) = (2k_2 r + l_2)(m - ab) + l_1 b,$$

$$\delta_1 = (\varepsilon_1 + \varepsilon_2 a) \bmod 2, \qquad \delta_2 = (\varepsilon_1 b + \varepsilon_2(m - ab)) \bmod 2,$$

and R *is a complete residue system* mod m.

(In (12), the expression $x \bmod 2$ denotes the remainder—0 or 1–upon dividing x by 2.) We should point out that whenever this formula is used, all the terms in the expansion have the same K_1, K_2 pair and the same δ_1, δ_2 pair, since these quantities are not functions of r. These invariant numbers are central to the uses we will make of this formula in the proofs that follow. In particular, they show that a T^2 equation transforms into a T^2 equation. (Whenever we list the parameters a, b, m in this expansion, we will write them together in the notation $[a, b, m]$.)

After using the expansion formula, which has m terms, it is important to reduce the terms so they can be properly combined.

Definition. If $\varepsilon \in \{0, 1\}$, then the function $T_\varepsilon(k, l)$ is said to be **reduced**, or in **reduced form**, if $0 \le l \le k$.

Because of the property (cf. [2, (14)])

$$(13) \qquad T_\varepsilon(k, -l) = T_\varepsilon(k, l), \qquad \varepsilon \in \{0, 1\},$$

we can assume $l \ge 0$. (For T_2 and T_3 we have the formula $T_2(k, -l) = T_3(k, l)$.) If $l > k$, we can reduce the value in the second argument by applying the following single-step reduction formula and (13)—over and over if necessary—until T is in reduced form.

Reduction Formula. *For* $k, l \in Z^+$, $k < l$, *and* $\varepsilon \in \{0, 1\}$,

$$(14) \qquad x^r T_\varepsilon(k, l) = (-1)^\varepsilon x^{r-(l-k)} T_\varepsilon(k, 2k - l).$$

Proof. From (2) we have

$$\begin{aligned} x^r T_\varepsilon(k, l) &= x^r \sum_{-\infty}^{\infty} (-1)^{\varepsilon n} x^{kn^2 + ln} = x^r \sum_{-\infty}^{\infty} (-1)^{\varepsilon(-n-1)} x^{k(-n-1)^2 + l(-n-1)} \\ &= (-1)^\varepsilon x^{r-(l-k)} \sum_{-\infty}^{\infty} (-1)^{\varepsilon n} x^{kn^2 + (2k-l)n} \\ &= (-1)^\varepsilon x^{r-(l-k)} T_\varepsilon(k, 2k - l). \quad \square \end{aligned}$$

This kind of reduction was used previously in [2, p. 309], where it was worked out separately in each case. (The same reduction formula holds for T_2 and T_3.)

Lemma 1. *If* $\varepsilon \in \{0, 1\}$, *then*

$$T_\varepsilon(k, l) = T(4k, 2l) + (-1)^\varepsilon x^{k-l} T(4k, 4k-2l).$$

Proof. This expansion comes from splitting the index values in the sum on the left into even and odd parts. □

Also, from [2, (15)] we have

Lemma 2. *We have* $T_1(\frac{k}{2}, r\frac{k}{2}) = 0$, *where* k *is a positive integer and* r *is an odd integer.*

We are now in a position to give the first proof of (9).

First Proof of Theorem 2. The proof falls naturally into two parts.
Part 1. Our purpose here is to use the expansion formula to transform the terms of equation (9) equivalently into equation (16), an equation that involves only the Q-function defined in (10).

We first transform the two terms on the left of (9) by the same expansion with the parameters [1,1,8]. (Note that (11) is satisfied.) This gives two groups of eight $T \cdot T$ terms, so the left-hand side of (9) with the T's reduced becomes

$$\begin{aligned} &x^{16}T(12, 10)T(84, 70) + x^9 T(12, 11)T(84, 49) + x^3 T(12, 8)T(84, 28) \\ &\quad + T(12, 5)T(84, 7) + T(12, 2)T(84, 14) + x^3 T(12, 1)T(84, 35) \\ &\quad + x^9 T(12, 4)T(84, 56) + x^{18} T(12, 7)T(84, 77) \\ &\quad - x^2[x^{12} T(12, 2)T(84, 70) + x^5 T(12, 5)T(84, 49) \\ &\qquad + xT(12, 8)T(84, 28) + T(12, 11)T(84, 7) \\ &\qquad + T(12, 10)T(84, 14) + x^2 T(12, 7)T(84, 35) \\ &\qquad\qquad + x^7 T(12, 4)T(84, 56) + x^{15} T(12, 1)T(84, 77)]. \end{aligned}$$

Since the third and seventh terms in the first group respectively cancel the corresponding terms in the second group, we get the equation

$$\begin{aligned} &T(\tfrac{21}{2}, \tfrac{7}{2})T(\tfrac{3}{2}, \tfrac{3}{2}) - x^2 T(\tfrac{21}{2}, \tfrac{21}{2})T(\tfrac{3}{2}, \tfrac{1}{2}) \\ &\quad = T(12, 2)T(84, 14) + T(12, 5)T(84, 7) \\ &\qquad + x^3 T(12, 1)T(84, 35) + x^9 T(12, 11)T(84, 49) \\ &\qquad + x^{16} T(12, 10)T(84, 70) + x^{18} T(12, 7)T(84, 77) \\ &\qquad - x^2 T(12, 10)T(84, 14) - x^2 T(12, 11)T(84, 7) \\ &\qquad - x^4 T(12, 7)T(84, 35) - x^7 T(12, 5)T(84, 49) \\ &\qquad - x^{14} T(12, 2)T(84, 70) - x^{17} T(12, 1)T(84, 77). \end{aligned} \tag{15}$$

Rearranging and grouping the terms on the right of (15), we obtain

$$\begin{aligned}
&T(\tfrac{21}{2}, \tfrac{7}{2})T(\tfrac{3}{2}, \tfrac{3}{2}) - x^2 T(\tfrac{21}{2}, \tfrac{21}{2})T(\tfrac{3}{2}, \tfrac{1}{2}) \\
&\quad = [T(12, 5)T(84, 7) - x^2 T(12, 11)T(84, 7) - x^7 T(12, 5)T(84, 49) \\
&\qquad\qquad + x^9 T(12, 11)T(84, 49)] \\
&\qquad + x^3[T(12, 1)T(84, 35) - xT(12, 7)T(84, 35) \\
&\qquad\qquad - x^{14}T(12, 1)T(84, 77) + x^{15}T(12, 7)T(84, 77)] \\
&\qquad + [T(12, 2)T(84, 14) - x^2 T(12, 10)T(84, 14) \\
&\qquad\qquad - x^{14}T(12, 2)T(84, 70) + x^{16}T(12, 10)T(84, 70)] \\
&\quad = [T(12, 5) - x^2 T(12, 11)][T(84, 7) - x^7 T(84, 49)] \\
&\qquad + x^3[T(12, 1) - xT(12, 7)][T(84, 35) - x^{14}T(84, 77)] \\
&\qquad + [T(12, 2) - x^2 T(12, 10)][T(84, 14) - x^{14}T(84, 70)] \\
&\quad = Q(8, 3)Q(56, 7) + x^3 Q(8, 1)Q(56, 21) + T_1(3, 1)T_1(21, 7).
\end{aligned}$$

(Here, (10) was used on the first four brackets, and Lemma 1 on the last two.) But the T_1 's in the third term are also Q 's, since $T_1(3, 1) = (0)_2 = Q(8, 2)$, and so $T_1(21, 7) = (0)_{14} = Q(56, 14)$ (by $x \to x^7$). If we now equate this expression to the left-hand side of (9) (which equals $2Q(8, 2)Q(56, 14)$), we arrive at the remarkable Q^2 equation

$$(16) \qquad Q(8, 3)Q(56, 7) + x^3 Q(8, 1)Q(56, 21) = Q(8, 2)Q(56, 14).$$

Part 2. Our next purpose is to transform (16) into (17) and then to prove (17) by using the expansion formula and an expansion of zero.

To begin, write the six Q 's in (16) in their product form using (10), viz.

$$\begin{aligned}
&Q(8, 3) = (0)_8(\pm 2)_{16}(\pm 3)_8 = (\pm 1)_8(\pm 3)_8 \cdot (0)_8[\pm 1]_8, \\
&Q(8, 1) = (0)_8(\pm 6)_{16}(\pm 1)_8 = (\pm 1)_8(\pm 3)_8 \cdot (0)_8[\pm 3]_8, \\
&Q(56, 7) = (\pm 7)_{56}(\pm 21)_{56} \cdot (0)_{56}[\pm 21]_{56}, \\
&Q(56, 21) = (\pm 7)_{56}(\pm 21)_{56} \cdot (0)_{56}[\pm 7]_{56}, \\
&Q(8, 2) = (0)_2 \text{ and } Q(56, 14) = (0)_{14}.
\end{aligned}$$

Substituting these products into (16), dividing by $(\pm 1, \pm 3)_8(\pm 7, \pm 21)_{56}$, and using (3) gives

$$T(4, 3)T(28, 7) + x^3 T(4, 1)T(28, 21) = \frac{(0)_2(0)_{14}}{(\pm 1, \pm 3)_8(\pm 7, \pm 21)_{56}} = \frac{(0)_2(0)_{14}}{(1)_2(7)_{14}}.$$

By (4), we have $\frac{(0)_2}{(1)_2} = \frac{1}{2}T(\frac{1}{2}, \frac{1}{2})$ and $\frac{(0)_{14}}{(7)_{14}} = \frac{1}{2}T(\frac{7}{2}, \frac{7}{2})$. Substituting these into the term on the far right, we obtain the new equation

$$(17) \qquad 4T(4, 3)T(28, 7) + 4x^3 T(4, 1)T(28, 21) = T(\tfrac{1}{2}, \tfrac{1}{2})T(\tfrac{7}{2}, \tfrac{7}{2}).$$

Note that the terms on the left of (17) have the k-pair $k_1 = 4$, $k_2 = 28$, while the term on the right has $k_1 = \frac{1}{2}$, $k_2 = \frac{7}{2}$. This difference suggests the following question: Can the right-hand side of (17) be expanded to produce terms with the k-pair (4,28) or (28,4)? The answer is "yes"; for if we take $k_1 = \frac{7}{2}$, $k_2 = \frac{1}{2}$ with the parameters [1,1,8], we find that

$$T(\tfrac{7}{2}, \tfrac{7}{2})T(\tfrac{1}{2}, \tfrac{1}{2}) = \sum_{r=-4}^{3} x^{\frac{1}{2}(r^2+r)} T(4, 3-r)T(28, 7r+7), \tag{18}$$

or after reducing,

$$\begin{aligned} T(\tfrac{7}{2}, \tfrac{7}{2})T(\tfrac{1}{2}, \tfrac{1}{2}) = {} & 2T(4, 3)T(28, 7) + 2x^3 T(4, 1)T(28, 21) \\ & + 2xT(4, 2)T(28, 14) + x^6 T(4, 0)T(28, 28) \\ & + T(4, 4)T(28, 0). \end{aligned} \tag{19}$$

Equating the left-hand side of (17) to the right-hand side of (19) gives

$$\begin{aligned} 2T(4, 3)T(28, 7) + 2x^3 T(4, 1)T(28, 21) - 2xT(4, 2)T(28, 14) \\ - x^6 T(4, 0)T(28, 28) - T(4, 4)T(28, 0) = 0. \end{aligned} \tag{20}$$

Now observe that the terms on the left of (20) are the same as those on the right of (19) except for the minus signs. Also note that the negative terms occur in the sum in (18) at the odd values of r. Thus, (20) can be written as the alternating sum

$$\sum_{r=-4}^{3} (-1)^r x^{\frac{1}{2}(r^2+r)} T(4, 3-r)T(28, 7r+7) = 0. \tag{21}$$

But this equation is valid since the left-hand side results from applying the expansion theorem to $T_1(\frac{7}{2}, \frac{7}{2})T_1(\frac{1}{2}, \frac{1}{2})$ with parameters [1,1,8], and the latter product is zero by Lemma 2. This establishes equation (9). □

Remark. In the rest of this paper the phrase "expansion of zero" will be used to designate a linear sum of T^2 terms, say $f(x)$, which arise by applying the expansion theorem to a single TT term which is identically zero. Then the T^2 equation $f(x) = 0$ holds. Thus, the left-hand side of (21) is an example of an expansion of zero.

4. An extension of the Expansion Formula

We begin this section by stating an important property of $T_3(k, l)$.

Lemma 3. *We have* $T_3(\frac{k}{2}, r\frac{k}{2}) = 0$, *if* k *is a positive integer and* $r \equiv 1 \pmod 4$.

Proof. The argument is essentially the same as that given in [2, (15)]. □

We will also use this result as part of another expansion of zero. The problem with doing this, however, is that the expansion formula in §3 only gives expansions for the four products $T_{\varepsilon_1}T_{\varepsilon_2}$, where ε_1, $\varepsilon_2 \in \{0, 1\}$. Thus, to be able to have an expansion of zero involving T_2 or T_3, we must consider whether there is an expansion formula for $T_{\varepsilon_1}T_{\varepsilon_2}$, when ε_1, $\varepsilon_2 \in \{0, 1, 2, 3\}$. If one examines the other 12 of the 16 possible products, it becomes clear that there is no expansion in some cases. Since further conditions on the parity of a, b, and m (and in some cases even more complicated conditions) are required for other expansion formulas to exist, we will not go into this more general question here. Instead we will confine our discussion to the case we will actually use in the second proof of (9).

Theorem 4. *Let a, b, and $m \in Z^+$, where $a \equiv 2 \pmod 4$, $b \equiv 1 \pmod 2$, and $4 \mid m$. Also, let (k_1, l_1), $(k_2, l_2) \in \{(\frac{i}{2}, \frac{j}{2}) : (i, j) \in Z^+ \times Z, i \equiv j \pmod 2\}$. If the separability condition*

$$k_1 b = k_2 a(m - ab)$$

is satisfied, then

$$T_1(k_1, l_1)T_3(k_2, l_2) = \sum_{r \in R} (-1)^{r(r-1)/2} x^{k_2 r^2 + l_2 r} T(K_1, L_1(r)) T(K_2, L_2(r)),$$

where

$$K_1 = k_1 + k_2 a^2, \qquad K_2 = k_2 m(m - ab),$$

$$L_1(r) = l_1 - l_2 a - 2k_2 ar, \qquad L_2(r) = (2k_2 r + l_2)(m - ab) + l_1 b,$$

and R is a complete residue system mod m.

Proof. The proof is substantially the same as the proof in [2, pp. 306–307]. Here, however, the exponent of (-1) in the definition of T_3 is quadratic in n, so the way it transforms when the variables are changed in the proof is more complicated than before. In particular, we have

$$\begin{aligned} T_1(k_1, l_1)T_3(k_2, l_2) &= \sum_i (-1)^i x^{k_1 i^2 + l_1 i} \sum_j (-1)^{j(j-1)/2} x^{k_2 j^2 + l_2 j} \\ &= \sum_{i,j} (-1)^{j(j-1)/2 + i} x^{k_1 i^2 + l_1 i + k_2 j^2 + l_2 j}. \end{aligned}$$

After carrying out the three transformations in the proof (viz. $j = n - ai$, $n = sm + r$, and $i = t + bs$) on just the exponent of (-1) (the rest is the same as before), we find that the sign in the final sum is determined by

$$(-1)^{t + bs + \frac{1}{2}[r + (m-ab)s - at][r - 1 + (m-ab)s - at]}. \tag{22}$$

Separating this power into its r, s, t, and st factors, we find this is

$$(-1)^{r(r-1)/2}\cdot(-1)^{bs+(m-ab)rs-\frac{1}{2}(m-ab)s+\frac{1}{2}(m-ab)^2s^2}\cdot(-1)^{t-art+\frac{1}{2}at+\frac{1}{2}a^2t^2}\cdot(-1)^{-a(m-ab)st}.$$

Since a is even, the fourth factor equals 1. Also, since b is odd and $m-ab\equiv 2 \pmod 4$, then

$$(-1)^{bs+(m-ab)rs-\frac{1}{2}(m-ab)s+\frac{1}{2}(m-ab)^2s^2}=(-1)^{s+0+s+0}=1,$$

and

$$(-1)^{t-art+\frac{1}{2}at+\frac{1}{2}a^2t^2}=(-1)^{t+0+t+0}=1.$$

Thus, the sign in the sum indexed by r is given by $(-1)^{r(r-1)/2}$ while the signs in the sums indexed by s and t are all plus, so the expansion contains only T_0's. □

5. The second proof of Theorem 2

We begin this section by introducing some useful terminology.

Definition. A T^2 set, or a T^2 equation, will be called *balanced* if the (k_1, k_2)-pair in each of its terms is the same. We will also say that the set or equation is balanced "at (k_1, k_2)".

For example, equation (20) is balanced at (4, 28). By analogy, one might also say that equation (16) is a "Q^2 equation balanced at (8, 56)".

In this section we will prove (9) without first transforming it into a Q^2 equation. Nonetheless, we will still use the two ideas in the proof of (17), viz. (i) the expansion formula is employed to change a T^2 set into a balanced T^2 set; (ii) this balanced set is then shown to be zero by grouping its terms into subsums, each of which is an expansion of zero. Although the ideas in this proof are the same as in the first proof, the new proof is more complicated because of the large number of terms that must be dealt with.

Second Proof of Theorem 2. Our goal here is to verify equation (9) written as

$$(23)\qquad T(\tfrac{21}{2},\tfrac{7}{2})T(\tfrac{3}{2},\tfrac{3}{2})-x^2T(\tfrac{21}{2},\tfrac{21}{2})T(\tfrac{3}{2},\tfrac{1}{2})-2T_1(21,7)T_1(3,1)=0.$$

The first step is the same as the beginning of the proof of (15): Expand $T(\frac{21}{2},\frac{7}{2})T(\frac{3}{2},\frac{3}{2})$ and $x^2T(\frac{21}{2},\frac{21}{2})T(\frac{3}{2},\frac{1}{2})$ with [1, 1, 8] and cancel terms in $T(\frac{21}{2},\frac{7}{2})T(\frac{3}{2},\frac{3}{2})-x^2T(\frac{21}{2},\frac{21}{2})T(\frac{3}{2},\frac{1}{2})$ to obtain (15). Next, expand all but the first and fifth terms on the right side of (15) with $k_1=84$, $k_2=12$ and [1, 1, 8]. Expand the first and fifth terms in (15), i.e., $T(84, 14)T(12, 2)$ and $x^{16}T(84, 70)T(12, 10)$, with $k_1=84$, $k_2=12$ and [7, 1, 8]. (For convenience, we write the factor $T(96, *)$ first in the $T\cdot T$ pairs in the second expansion.) We now write $A_i(x)$ and $B_i(x)$, $1\le i\le 48$, for the surviving terms in the expansions of $T(\frac{21}{2},\frac{7}{2})T(\frac{3}{2},\frac{3}{2})$ and $-x^2T(\frac{21}{2},\frac{21}{2})T(\frac{3}{2},\frac{1}{2})$, respectively.

Here, each of the 48 A_i-terms (all are positive) and 48 B_i-terms (all are negative) have the form $x^{\alpha_i}T(96, L_{1i})T(672, L_{2i})$. (These are listed in Tables 1 and 2 respectively.) Expanding the third term, $-2T_1(21, 7)T_1(3, 1)$, in (23) with [5, 5, 32] gives the $G_i(x)$ terms of the form $\pm x^{\alpha_i}T(96, L_{1i})T(672, L_{2i})$ listed in Table 3. (Each of these appears twice because of the coefficient 2.)

At this point we have 160 ($= 48 + 48 + 2 \cdot 32$) terms in the expansion, viz.

$$\begin{aligned} &T(\tfrac{21}{2}, \tfrac{7}{2})T(\tfrac{3}{2}, \tfrac{3}{2}) - x^2 T(\tfrac{21}{2}, \tfrac{21}{2})T(\tfrac{3}{2}, \tfrac{1}{2}) - 2T_1(21, 7)T_1(3, 1) \\ &\quad = \sum_{i=1}^{48} A_i(x) + \sum_{i=1}^{48} B_i(x) + 2\sum_{i=1}^{32} G_i(x). \end{aligned} \tag{24}$$

Thirty-two of these positive and negative G_i's cancel with the (positive) A_i's and the (negative) B_i's. (The terms that cancel are indicated by 'C' in Tables 1 – 3. In Table 3, those G's numbered $G_1, \dots, G_{16}$ cancel once and those numbered $G_{17}, \dots, G_{24}$ cancel twice.) This leaves 96 uncanceled terms in the three tables. We next group these terms into three classes labeled "D", "E", and "F". The sum of the terms in each of these three classes is zero, because each sum is an expansion of zero. In particular, the expansions of $-2x^2T_1(42, 14)T_1(6, 6)$ with [3, 3, 16], as well as $-T_1(42, 7)T_3(\frac{3}{2}, \frac{3}{2})$ and $x^7T_1(42, 35)T_3(\frac{3}{2}, \frac{3}{2})$ with [6, 3, 32], exactly give the groups D, E and F respectively, i.e.,

$$-2x^2T_1(42, 14)T_3(6, 6) = 2\sum_{i=1}^{16} D_i(x),$$

$$-T_1(42, 7)T_3(\tfrac{3}{2}, \tfrac{3}{2}) = \sum_{i=1}^{32} E_i(x),$$

and

$$x^7T_1(42, 35)T_3(\tfrac{3}{2}, \tfrac{3}{2}) = \sum_{i=1}^{32} F_i(x).$$

(Note that each of the terms denoted by D appears twice in Tables 2 and 3.) Thus, we find that

$$\begin{aligned} &T(\tfrac{21}{2}, \tfrac{7}{2})T(\tfrac{3}{2}, \tfrac{3}{2}) - x^2 T(\tfrac{21}{2}, \tfrac{21}{2})T(\tfrac{3}{2}, \tfrac{1}{2}) - 2T_1(21, 7)T_1(3, 1) \\ &\quad = 2\sum_{i=1}^{16} D_i(x) + \sum_{i=1}^{32} E_i(x) + \sum_{i=1}^{32} F_i(x) \\ &\quad = -2x^2T_1(42, 14)T_1(6, 6) - T_1(42, 7)T_3(\tfrac{3}{2}, \tfrac{3}{2}) \\ &\qquad + x^7T_1(42, 35)T_3(\tfrac{3}{2}, \tfrac{3}{2}) = 0, \end{aligned}$$

since each term in the final trinomial is zero by Lemmas 2 and 3 with $r = 1$. This establishes (9). □

TABLE 1. $A_i(x) = x^{\alpha_i} T(96, L_{1i}) T(672, L_{2i})$

All the terms in this table are positive

$T(84, 14)T(12, 2)$, [7, 1, 8]

i	type	α_i	L_{1i}	L_{2i}
1.	E	0	16	0
2.	F	10	8	168
3.	F	14	40	168
4.	F	44	32	336
5.	F	52	64	336
6.	F	102	56	504
7.	F	114	88	504
8.	E	184	80	672

$T(84, 7)T(12, 5)$, [1, 1, 8]

i	type	α_i	L_{1i}	L_{2i}
9.	C	0	2	42
10.	F	7	26	126
11.	F	17	22	210
12.	C	38	50	294
13.	C	58	46	378
14.	F	93	74	462
15.	F	123	70	546
16.	C	170	94	630

$x^3 T(84, 35)T(12, 1)$, [1, 1, 8]

i	type	α_i	L_{1i}	L_{2i}
17.	E	3	34	42
18.	C	14	58	126
19.	C	16	10	210
20.	E	49	82	294
21.	E	53	14	378
22.	C	98	86	462
23.	C	114	38	546
24.	E	157	62	630

$x^9 T(84, 49)T(12, 11)$, [1, 1, 8]

i	type	α_i	L_{1i}	L_{2i}
25.	E	9	38	126
26.	C	10	62	42
27.	C	32	14	294
28.	E	35	86	210
29.	C	70	82	378
30.	E	79	10	462
31.	E	119	58	546
32.	C	150	34	630

$x^{16} T(84, 70)T(12, 10)$, [7, 1, 8]

i	type	α_i	L_{1i}	L_{2i}
33.	F	16	80	0
34.	E	18	56	168
35.	E	30	88	168
36.	E	44	32	336
37.	E	52	64	336
38.	E	94	8	504
39.	E	98	40	504
40.	F	168	16	672

$x^{18} T(84, 77)T(12, 7)$, [1, 1, 8]

i	type	α_i	L_{1i}	L_{2i}
41.	C	18	70	126
42.	F	23	94	42
43.	C	30	74	210
44.	F	37	46	294
45.	F	59	50	378
46.	C	80	22	462
47.	C	112	26	546
48.	F	147	2	630

TABLE 2. $B_i(x) = x^{\alpha_i} T(96, L_{1i}) T(672, L_{2i})$

All the terms in this table are negative

$x^2 T(84, 14) T(12, 10)$, [1, 1, 8]

i	type	α_i	L_{1i}	L_{2i}
1.	D	2	4	84
2.	C	4	28	84
3.	C	24	20	252
4.	D	30	52	252
5.	D	70	44	420
6.	C	80	76	420
7.	C	140	68	588
8.	D	150	92	588

$x^2 T(84, 7) T(12, 11)$, [1, 1, 8]

i	type	α_i	L_{1i}	L_{2i}
9.	D	2	4	84
10.	E	3	20	84
11.	F	25	28	252
12.	C	28	44	252
13.	C	72	52	420
14.	F	77	68	420
15.	E	143	76	588
16.	D	150	92	588

$x^4 T(84, 35) T(12, 7)$, [1, 1, 8]

i	type	α_i	L_{1i}	L_{2i}
17.	C	4	28	84
18.	E	9	52	84
19.	F	23	4	252
20.	D	38	76	252
21.	D	66	20	420
22.	F	87	92	420
23.	E	133	44	588
24.	C	140	68	588

$x^7 T(84, 49) T(12, 5)$, [1, 1, 8]

i	type	α_i	L_{1i}	L_{2i}
25.	F	7	44	84
26.	D	14	68	84
27.	C	24	20	252
28.	E	45	92	252
29.	C	80	76	420
30.	E	65	4	420
31.	D	130	28	588
32.	F	135	52	588

$x^{14} T(84, 70) T(12, 2)$, [1, 1, 8]

i	type	α_i	L_{1i}	L_{2i}
33.	D	14	68	84
34.	C	24	92	84
35.	C	28	44	252
36.	D	38	76	252
37.	D	66	20	420
38.	C	72	52	420
39.	C	128	4	588
40.	D	130	28	588

$x^{17} T(84, 77) T(12, 1)$, [1, 1, 8]

i	type	α_i	L_{1i}	L_{2i}
41.	F	17	76	84
42.	C	24	92	84
43.	D	30	52	252
44.	E	35	68	252
45.	E	67	28	420
46.	D	70	44	420
47.	C	128	4	588
48.	F	129	20	588

TABLE 3. $G_i(x) = \pm x^{\alpha_i} T(96, L_{1i}) T(672, L_{2i}) - 2T_1(21, 7)T_1(3, 1)$, [5, 5, 32]

Each term occurs twice

Negative Terms					Positive Terms				
i	type	α_i	L_{1i}	L_{2i}	i	type	α_i	L_{1i}	L_{2i}
1.	CE	0	2	42	17.	CC	4	28	84
2.	CF	10	62	42	18.	CC	24	20	252
3.	CF	14	58	126	19.	CC	24	92	84
4.	CF	16	10	210	20.	CC	28	44	252
5.	CE	18	70	126	21.	CC	72	52	420
6.	CE	30	74	210	22.	CC	80	76	420
7.	CF	32	14	294	23.	CC	140	68	588
8.	CE	38	50	294	24.	CC	128	4	588
9.	CE	58	46	378	25.	DD	2	32	0
10.	CF	70	82	378	26.	DD	14	40	168
11.	CE	80	22	462	27.	DD	30	88	168
12.	CF	98	86	462	28.	DD	42	16	336
13.	CE	112	26	546	29.	DD	58	80	336
14.	CF	114	38	546	30.	DD	94	8	504
15.	CF	150	34	630	31.	DD	102	56	504
16.	CE	170	94	630	32.	DD	178	64	672

6. COMMENTARY ON THE SECOND PROOF

The problem of putting together the second proof was solved by first finding what expansions could be used on the left of (23) and then discovering how the resulting terms could be grouped into subsums that were expansions of zero. To help in analyzing this problem, we devised two algorithms, *Forward* and *Backward.*

The variables k_1, k_2, K_1, and K_2 used in both these algorithms are positive multiples of $\frac{1}{2}$. When x and y are rational, the notation $x \mid y$ means $\frac{y}{x}$ is an integer.

Algorithm *Forward.* Given k_1, k_2, and bound M. Find all values of K_1, K_2, a, b, and $m \le M$ that satisfy (11) and the first two equations in (12).

for $m = 2$ to M do
  for $a = 1$ to $m - 1$ do
    $K_1 = k_1 + k_2 a^2$
    if $K_1 \mid a k_2 m$ then
      $b = \frac{a k_2 m}{K_1}$, $K_2 = k_2 m(m - ab)$
    output K_1, K_2, a, b, m

Proof. To verify the algorithm, we need only show that (11) is satisfied:

$$b = \frac{ak_2 m}{K_1} \implies b(k_1 + k_2 a^2) = ak_2 m \implies k_1 b = k_2 a(m - ab). \quad \square$$

The *Backward* program is somewhat more elaborate, but needs no bound, because, given K_1 and K_2, the formulas in (12) clearly show that only a finite number of positive integers can satisfy them.

Algorithm *Backward.* Given K_1, K_2. Find all possible values of k_1, k_2, a, b, m that satisfy (11) and the first two equations in (12).

for a such that $1 \le a^2 < K_1$ do
  for k_2 such that $\frac{1}{2} \le a^2 k_2 < K_1$ and $k_2 \mid K_2$ * do
    $k_1 = K_1 - k_2 a^2$
    if $\frac{k_2 K_2}{k_1 K_1} = s^2$, a perfect square, then
      $b = as$, $m = \frac{aK_2}{bk_1}$
        output k_1, k_2, a, b, m

Proof. Since $K_1 = k_1 + k_2 a^2$ and $k_1, k_2 > 0$, the right-hand side in this equation is > 0, so there are at most a finite number of values for k_1, k_2 and a for which the equation holds. We must show that the numbers k_1, k_2, a, b, and m satisfy the required conditions. There is nothing to prove for k_1, considering how it was obtained. To show (11): $b^2 = \frac{a^2 k_2 K_2}{k_1 K_1} \implies bK_1 = \frac{a^2 k_2 K_2}{bk_1} = ak_2 m$. Thus,

$$b(k_1 + k_2 a^2) = ak_2 m \implies k_1 b = k_2 a(m - ab).$$

The second equation in (12) now follows, since

$$bk_1 m = ak_2 m(m - ab) \implies k_2 m(m - ab) = \frac{bk_1 m}{a} = K_2. \quad \square$$

Used together, these programs allow the tracing of composition "paths"—starting with (k_1, k_2) and ending at (K_1, K_2)—through all possible intermediate pairs, which are discovered by using *Backward* on (K_1, K_2). Our first problem is to turn the T^2 equation (23) into a balanced equation. The question is: At what K-pair should the equation be balanced? In the first proof we were able to expand just the right-hand side of (17) to balance the equation at (4, 28). This is the first possibility to consider here: Starting with $k_1 = \frac{21}{2}$, $k_2 = \frac{3}{2}$ (or $k_1 = \frac{3}{2}$, $k_2 = \frac{21}{2}$), can we expand to $K_1 = 21$, $K_2 = 3$, the K-pair of the third term? Using (12), we would have to have either

$$21 = \frac{21}{2} + \frac{3}{2}a^2 \implies a^2 = 7 \implies a \notin Z^+$$

*For example, if $K_2 = 15$, then the positive values of k_2 such that $k_2 \mid K_2$ are $1, 3, 5, 15, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{15}{2}$.

or

$$21 = \frac{3}{2} + \frac{21}{2}a^2 \implies a^2 = \frac{13}{7} \implies a \notin Z^+.$$

Thus, an expansion of just the first two terms of (23) will not work. We therefore conclude that all the terms will have to be expanded to terms with a common K-pair if equation (23) is to be expanded to a balanced equation. The next question is: What are the possible common K-pairs? To determine this, we used *Forward* to find all expansions of $(\frac{21}{2}, \frac{3}{2})$ with their parameters for $2 \le m \le 64$ (the complete set of values are listed in Table 4A only for $2 \le m \le 32$ and $m = 64$) and of (21, 3) with their parameters for $2 \le m \le 32$ (these are given in Table 4B). These tables show very well just how restrictive condition (11) is. Actually, it would be important (and remarkable) if there were always a balanced equation to which one could expand from a T^2 equation in which all the terms cancel directly, making any further proof unnecessary (cf. [2, (37)]).

TABLE 4A. *Forward*

$k_1 = \frac{21}{2}, \quad k_2 = \frac{3}{2}, \quad 2 \le m \le 64$

K_1	K_2	a	b	m
→ 12	84	1	1	8
→ 84	12	7	1	8
33/2	231/2	2	2	11
12	336	1	2	16
*24	168	3	3	16
84	48	7	2	16
33/2	462	2	4	22
69/2	483/2	4	4	23
12	756	1	3	24
84	108	7	3	24
609/2	87/2	14	2	29
12	1344	1	4	32
*24	672	3	6	32
*48	336	5	5	32
84	192	7	4	32
⋮	⋮	⋮	⋮	⋮
*24	2688	3	12	64
*48	1344	5	10	64
84	768	7	8	64
*672	96	21	3	64

TABLE 4B. *Forward*

$k_1 = 21, \quad k_2 = 3, \quad 2 \le m \le 32$

K_1	K_2	a	b	m
*24	168	1	1	8
*168	24	7	1	8
33	231	2	2	11
*24	672	1	2	16
*48	336	3	3	16
168	96	7	2	16
33	924	2	4	22
69	483	4	4	23
24	1512	1	3	24
168	216	7	3	24
609	87	14	2	29
*24	2688	1	4	32
*48	1344	3	6	32
*96	672	5	5	32
168	384	7	4	32

TABLE 5. *Backward*

$K_1 = 672 \quad K_2 = 96$

k_1	k_2	a	b	m
→ 84	12	1	1	8
→ 12	84	1	7	8
84	3	2	1	16
12	21	2	7	16
42	6	3	3	16
3/2	21/2	3	21	64
21	3	5	5	32
42	3/2	6	3	32

$K_1 = 96 \quad K_2 = 672$

k_1	k_2	a	b	m
→ 84	12	7	1	8
84	3	14	1	16
21/2	3/2	21	3	64

An examination of the k-pairs that are common to Tables 4A and 4B (indicated by an asterisk) shows that the smallest such pair is (24, 168) (or possibly (168, 24)), which occurs with [3,3,16] in Table 4A and with both [1,1,8] and [7,1,8] in Table 4B. Thus, there are two possible expansions to examine. To establish either of the resulting equations by means of expansions of zero, we must use *Backward* to find all the k-pair "ancestors" of $K_1 = 24$, $K_2 = 168$ (or $K_1 = 168$, $K_2 = 24$) to use in this expansion. For (24,168), there are the three ancestors:

$$k_1 = 21\,,\ k_2 = 3 \text{ with } [1\,,\,1\,,\,8]\,, \qquad k_1 = 3\,,\ k_2 = 21 \text{ with } [1\,,\,7\,,\,8]\,,$$

$$\text{and} \quad k_1 = \tfrac{21}{2}\,,\ k_2 = \tfrac{3}{2} \text{ with } [3\,,\,3\,,\,16].$$

For (168,24) there is only $k_1 = 21$, $k_2 = 3$ with [7,1,8]. After examining all the possible expansions of zero, we find that none of them gives a proof of (23).

Thus, we must try another, larger K-pair from Tables 4A and 4B. This time we chose $K_1 = 96$, $K_2 = 672$ because *Backwards* gives a large collection of ancestors from which we can try to construct a proof. Although m has the large value 64 for this K-pair in Table 4A, there is a second way to expand the first two terms, viz. by the composition of two expansions: the first from $(\frac{21}{2}, \frac{3}{2})$ to (12, 84) and the second from (12, 84) to (672, 96). (These entries are indicated by arrows in Tables 4A and 5.) Even though it would seem that 64 terms would also be produced by this composition, it happens that some terms cancel after the first expansion, so there are actually only 96 terms that must be dealt with. We expand the third term in (23) to (96, 672) using the parameters [5,5,32] in Table 4B. When all the terms are combined, the 96 remaining terms must then be examined to see if they can be grouped into expansions of zero.

It is sometimes the case in this examination that a certain subsum of terms stands out as a candidate for such a set, as in the present case where the doubled

(D) terms form a noticeable set. From Table 3 the sum of the D terms is

$$\begin{aligned} D^+ \overset{\text{def}}{=} &(2, 32, 0) + (14, 40, 168) + (30, 88, 168) \\ &+ (42, 16, 336) + (58, 80, 336) + (94, 8, 504) \\ &+ (102, 56, 504) + (178, 64, 672) \end{aligned} \tag{25}$$

and the sum in Table 2 is

$$\begin{aligned} D^- \overset{\text{def}}{=} &(2, 4, 84) + (30, 52, 252) + (70, 44, 420) \\ &+ (150, 92, 588) + (38, 76, 252) + (66, 20, 420) \\ &+ (14, 68, 84) + (130, 28, 588). \end{aligned} \tag{26}$$

(Here the triple (α, L_1, L_2) is an abbreviated notation for the term $x^\alpha T(96, L_1)T(672, L_2)$.) Since the D^- terms are negative in Table 2, the question is whether $D^+ - D^-$ is an expansion of zero, say based on T_1 (T_3 is the other possibility). If so, we might have

$$D^+ - D^- = \pm x^2 T_1(k_1, l_1)T_1(k_2, k_2) = 0, \tag{27}$$

with the sign to be determined later. If this were the case, it would then follow that $D^+ + D^- = x^2 T_0(k_1, l_1)T_0(k_2, k_2)$, i.e., the sum with the same terms, but now with all plus signs. Adding these equations gives

$$x^{-2}D^+ = \tfrac{1}{2}T_0(k_1, l_1)T_0(k_2, k_2) = T_0(k_1, l_1)T_0(4k_2, 2k_2),$$

using (4), i.e., the assumption that $D^+ - D^-$ vanishes because it is an expansion of zero of this kind implies that the power series for $x^{-2}D^+$ must factor into a $T_0 \cdot T_0$ product. The factorization of this power series up to degree 1000 by the greedy algorithm (cf. [2, p. 311]) indicates that $x^{-2}D^+$ is undoubtedly equal to $(0, \pm 48, \pm 96, \pm 144, 168)_{336}(0)_{168}[\pm 12, \pm 28, \pm 36, \pm 56, \pm 60]_{168}$, which is $(0)_{84}[\pm 28]_{84}(0)_{48}[\pm 12]_{48} = T_0(42, 6)T_0(24, 12)$. Thus, we conclude that $k_1 = 42$, $l_1 = 14$, and $k_2 = 6$. Expanding the $T_1 \cdot T_1$ product with [3, 3, 16] (these parameters being computed from (11) and (12)) determines that the sign in (27) is negative: $D^+ - D^- = -x^2 T_1(42, 14)T_1(6, 6)$.

Of course, if the form for the expansion of zero we tried in (27) were not correct, this would become apparent when D^+ was factored; for not only would the expected form on the right not occur, but rather, the factorization would undoubtedly be the product of binomials in which the sequence of exponents increased, apparently without bound.

In the general case, where no particular subsum is a candidate for an expansion of zero, we can first find all the ancestors of (K_1, K_2) and then examine all the possible expansions of zero to see if the full T^2 set can be written as a combination of some of them. This is how the last two expansions of zero in the proof were found—the first being the sum of the doubles.

The irregularities in the proof, such as using two different but related expansions in the second expansion of the composition, and using T_3 in an expansion of zero, were found to be necessary to match the signs of the terms so a proof could be made by this method.

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